**Math Quadrotomy**

Theorems

1. *b* ≺ *c* ≡ ¬(*c* ≺ *b* ∨ *b* = *c* ∨ *incomp(b, c*))

Proof:

¬(*c* ≺ *b* ∨ *b* = *c* ∨ *incomp(b, c*))

= ⟨Lemma 1⟩

¬(*c* ≼ *b* ∨ *incomp(b, c*))

= ⟨(14.47.1)⟩

¬(*c* ≼ *b* ∨ (¬(*b* ≼ *c*) ∧ ¬(*c* ≼ *b*)))

= ⟨ (3.44b)⟩

¬(*c* ≼ *b* ∨ ¬(*b* ≼ *c*))

= ⟨(3.47b)⟩

*b* ≼ *c* ∧ ¬(*c* ≼ *b*)

= ⟨Lemma 1, twice⟩

(*b* ≺ *c* ∨ *b* = *c*) ∧ ¬(*c* ≺ *b* ∨ *b* = *c*)

= ⟨Lemma 2⟩

*b* ≺ *c* ∧ ¬(*b* = *c*) ∧ ¬(*c* ≺ *b*)

= ⟨Lemma 3, with *p*, *q* := *b* ≺ *c*, ¬(*c* ≺ *b*). *b* ≺ *c* ⇒ ¬(*c* ≺ *b*) by the definition of asymmetry, because strict orders are transitive and irreflexive, and therefore asymmetric⟩

*b* ≺ *c* ∧ ¬(*b* = *c*)

= ⟨Lemma 3, with *p*, *q* := *b* ≺ *c*, ¬(*b* = *c*). *b* ≺ *c* ⇒ ¬(*b* = *c*) by Lemma 4 and the fact that ≺ is irreflexive⟩

*b* ≺ *c* //

1. *c* ≺ *b* ≡ ¬(*b* ≺ *c* ∨ *b* = *c* ∨ *incomp(b, c*))

The proof is identical to (a), but with *b*, *c* := *c*, *b*.

1. *b* = *c* ≡ ¬(*b* ≺ *c* ∨ *c* ≺ *b* ∨ *incomp(b, c*))

Proof:

¬(*b* ≺ *c* ∨ *c* ≺ *b* ∨ *incomp(b, c*))

= ⟨(3.26)⟩

¬(*b* ≺ *c* ∨ *c* ≺ *b* ∨ *incomp(b, c*) ∨ *incomp(b, c*))

= ⟨(14.48.2), twice⟩

¬(¬(*b* ≼ *c*) ∨ ¬(*c* ≼ *b*))

= ⟨(3.47a) with (3.12) double negation twice⟩

*b* ≼ *c* ∧ *c* ≼ *b*

= ⟨Lemma 5, with the fact that partial orders are antisymmetric and reflexive⟩

*b* = *c* //

1. *incomp(b, c*) ≡ ¬(*b* ≺ *c* ∨ *c* ≺ *b* ∨ *b* = *c*)

Proof:

¬(*b* ≺ *c* ∨ *c* ≺ *b* ∨ *b* = *c*)

= ⟨(3.26)⟩

¬(*b* ≺ *c* ∨ *c* ≺ *b* ∨ *b* = *c* ∨ *b* = *c*)

= ⟨Lemma 1, twice⟩

¬(*b* ≼ *c* ∨ *c* ≼ *b*)

= ⟨(3.47b)⟩

¬(*b* ≼ *c*) ∧ ¬(*c* ≼ *b*)

= ⟨(14.47.1)⟩

*incomp(b, c)* //

Lemmas

Lemma 1

*b* ≺ *c* ∨ *b* = *c* ≡ *b* ≼ *c*

where ≺ is the reflexive reduction of ≼.

Proof:

*b* ≺ *c* ∨ *b* = *c*

= ⟨(14.15.4)⟩

⟨*b*, *c*⟩ ∈ ≺ ∨ *b* = *c*

= ⟨(14.15.3)⟩

⟨*b*, *c*⟩ ∈ ≺ ∨ ⟨*b*, *c*⟩ ∈ *iB*

= ⟨(11.20)⟩

⟨*b*, *c*⟩ ∈ ≺ ∪ *iB*

= ⟨(14.49b) Caveat: this step is not entirely formal. Theorem 14.49b states only that ≺ ∪ *iB* is a partial order, not specifically that it is the partial order the reflexive reduction of which is ≺. This step is still correct, however; this theorem is simply stated in English and not as strongly as and precisely as it could be.⟩

⟨*b*, *c*⟩ ∈ ≼

= ⟨(14.15.4)⟩

*b* ≼ *c* //

Lemma 2

(*p* ∨ *q*) ∧ ¬(*q* ∨ *r*) ≡ *p* ∧ ¬*q* ∧ ¬*r*

Proof:

(*p* ∨ *q*) ∧ ¬(*q* ∨ *r*)

= ⟨(3.47b)⟩

(*p* ∨ *q*) ∧ ¬*q* ∧ ¬*r*

= ⟨(3.46)⟩

((*p* ∧ ¬*q*) ∨ (*q* ∧ ¬*q*)) ∧ ¬*r*

= ⟨(3.42)⟩

((*p* ∧ ¬*q*) ∨ *false*) ∧ ¬*r*

= ⟨(3.30)⟩

*p* ∧ ¬*q* ∧ ¬*r* //

Lemma 3

(*p* ⇒ *q*) ⇒ (*p* ∧ *q* ≡ *p*)

Proof:

*p* ∧ *q* ≡ *p*

= ⟨(3.60)⟩

*p* ⇒ *q*

⇐ ⟨(3.71) Reflexivity of ⇒ with *p* := *p* ⇒ *q*⟩

*p* ⇒ *q* //

Lemma 4

*ρ* is irreflexive ⇒ (*b* *ρ* *c* ⇒ ¬(*b* = *c*))

Proof:

*b* *ρ* *c* ⇒ ¬(*b* = *c*)

= ⟨(3.59)⟩

¬(*b* *ρ* *c)* ∨ ¬(*b* = *c*)

= ⟨(3.47a)⟩

¬(*b* *ρ* *c* ∧ *b* = *c*)

= ⟨(3.84a)⟩

¬(*b* *ρ* *b* ∧ *b* = *c*)

= ⟨Assume the antecedent *ρ* is irreflexive, or ¬(*b ρ* *b)* for all *b*, with double negation⟩

¬(¬*true* ∧ *b* = *c*)

= ⟨(3.8)⟩

¬(*false* ∧ *b* = *c*)

= ⟨(3.40)⟩

¬*false*

= ⟨(3.13)⟩

*true* //

Lemma 5

*ρ* is antisymmetric ∧ *ρ* is reflexive ⇒ (*b* *ρ* *c* ∧ *c* *ρ* *b* ≡ *b* = *c*)

Proof by (4.7) Mutual Implication:

1. *b* *ρ* *c* ∧ *c* *ρ* *b* ⇒ *b* = *c*

Proof:

*b* *ρ* *c* ∧ *c* *ρ* *b*

⇒ ⟨Assume the antecedent *ρ* is antisymmetric, or *b* *ρ* *c* ∧ *c* *ρ* *b* ⇒ *b* = *c* for all *b*, *c*⟩

*b* = *c* //

1. *b* = *c* ⇒ *b* *ρ* *c* ∧ *c* *ρ* *b*

Proof:

*b* *ρ* *c* ∧ *c* *ρ* *b*

= ⟨Assume the antecedent *b* = *c*, twice⟩

*b* *ρ* *b* ∧ *b* *ρ* *b*

= ⟨(3.38)⟩

*b* *ρ* *b*

= ⟨Assume the antecedent *ρ* is reflexive, or *b* *ρ* *b* for all *b*⟩

*true* //

Application

α.

β.

γ.

δ.

Proof by (4.4) Deduction:

= ⟨Assume conjunct α of the antecedent⟩

= ⟨(3.28) Excluded middle, ⟩

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Proof:

= ⟨Lemma 6, with . The antecedent is true by Theorems a-d⟩

= ⟨(3.39) Identity of ∧, ⟩

= ⟨Lemma 8, with ⟩

= ⟨Lemma 7, , with . The conjuncts α through δ of the antecedent are satisfied by Theorems a-d, while the conjunct is satisfied by Lemma 9⟩

= ⟨ Lemma 7, , with . The conjuncts α through δ of the antecedent are satisfied as in the previous step. Note that the expressions replacing and are swapped in this step⟩

= ⟨ Lemma 7, , with . The conjuncts α through δ of the antecedent are satisfied as in the previous step⟩

= ⟨Lemma 6, with . The antecedent is true by Theorems a-d⟩

= ⟨(3.30) Identity of ∨, ⟩

= ⟨Theorem d with , with (3.12) Double negation⟩

= ⟨Lemma 9⟩

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Lemma 6

Proof by (4.4) Deduction:

= ⟨(3.47a) De Morgan, twice⟩

= ⟨Assume conjunct α of the antecedent, , with (3.12) Double negation⟩

= ⟨(3.28) Excluded middle, , twice⟩

= ⟨(3.29) Identity of ∨, ⟩

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Lemma 7

Proof by (4.4) Deduction:

= ⟨(3.39) Identity of ∧, ⟩

= ⟨Assume conjunct of the antecedent⟩

= ⟨(3.47b) De Morgan, twice⟩

= ⟨Assume conjunct α of the antecedent, ⟩

= ⟨(3.40) Idempotency of ∧, ⟩

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Lemma 8

Proof:

= ⟨(3.52) Equivalence, ⟩

= ⟨(3.52) Equivalence⟩

= ⟨(3.46) Distributivity of ∧ over ∨, ⟩

= ⟨(3.52) Equivalence ⟩

= ⟨(3.10) Definition of , ⟩

= ⟨(3.46) Excusive or, ⟩

= ⟨(3.46) Distributivity of ∧ over ∨⟩

= ⟨(3.24) Symmetry of ∨⟩

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Lemma 9

, where refers to and being incomparable under .

Proof:

= ⟨(14.47.1) Definition, Incomparable, , where the partial order is ⟩

= ⟨(3.47b) De Morgan, ⟩

= ⟨(14.50) Definition, Total Order: A partial order over is called a *total* or *linear* order if , with (9.16) Metatheorem, P is a theorem iff is a theorem, and the fact that ≤ is a total order⟩

= ⟨(3.8) Definition of ⟩

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